A COMPARISON OF STOCHASTIC ASSET MODELS

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ABSTRACT

A number of actuarial models designed for simulating future economic and investment conditions have been published. We discuss eight such models developed in the United Kingdom. We discuss their features, including the series they cover, the number of innovation "drivers" they use, the output variables they produce, etc. We then compare their results numerically. Finally we discuss possible extensions to such simulation models, which might improve their applicability, including: the use of various initial conditions; using "neutralising parameters"; using a "select period"; using alternative innovations; simulating over shorter intervals; and "hyperising" the models.

KEYWORDS

Stochastic asset models, parameters, initial conditions, innovations, distributions, select period, fat-tailedness, hypermodels, Wilkie models, Random walk variant models, Smith model, TY model, Cairns model, Whitten-Thomas model.

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1 INTRODUCTION

1.1 A number of stochastic asset models have been developed by actuaries since the Maturity Guarantees Working Party first derived a model for the movement of prices of ordinary shares based on dividends and dividend yields in 1980 (Ford *et al*, 1980). Such models are also described as "investment models", but perhaps "economic models" would be a better term, since they often include models for inflation and wages.

1.2 Our purpose in this paper is to describe and compare a number of published models, to provide some comparison of the distributions that result from them, and to show ways in which many of them can be elaborated to suit the needs of potential users. We have restricted ourselves to models based on United Kingdom data; this excludes models produced by Thompson (1996) for South Africa, and various models proposed in Finland and Australia.

1.3 The models that we shall consider are:

(a) the Wilkie model, as described in Wilkie (1995);
(b) the ARCH variation of the Wilkie model, also described in Wilkie (1995)
(c) the random walk model with a Wilkie-style inflation model and normally distributed innovations, described in Smith (1996) and there attributed by him to Kemp; we refer to this as a random walk variant or "RWV lognormal" models;
(d) as (c), but with α-stable distributions for innovations, also as described by Smith (1996); we refer to this as the "RWV α-stable" model;
(e) Smith's jump diffusion model, as described in Smith (1996), and further explained by Huber (1998);
(f) the TY model, as described in Yakoubov, Teeger & Duval (1999);
(g) the Cairns model, as described in Cairns (1999a and 1999b)
(h) the Whitten & Thomas model, as described in Whitten & Thomas (2000).

1.4 We do not consider the Exley model, which appears to us to be only partially described in an Appendix to Dyson & Exley (1995). Smith (1996) implemented a version of this model, but the correspondence of that implementation to the original paper is not clear to us. This is perhaps not surprising, in that, in the case of both papers, this particular asset model was not the main topic under discussion. We would welcome further specification of this model.

1.5 In general, not all the published papers specify fully the authors' models and the parameter values used by them, although Smith (1996) gives programme source code, which greatly helps, although this simplified in the case of his own model, for example in regard to the initial conditions. We have been aided by this additional information from him, and from Cairns and Whitten & Thomas, whose help we gratefully acknowledge.

1.6 As we shall see, all these models have many similar features, but of course differ in a number of ways. There are other types of model which we refer to, particularly the random walk or logarithmic Brownian motion model for shares; however, this is closely represented by what we are calling the RWV lognormal model. There are also many models for interest rates, but most of these seem to be designed for the pricing of derivatives, whereas the main purpose of the actuarial models, except perhaps for the Smith model, is the modelling of long-term investment performance.

1.7 There are also, we understand, a number of unpublished models in use by actuaries and others in a number of firms. Naturally we cannot discuss such models, since even if we knew the details of them, their authors have chosen that their details should remain confidential.

2 BASIC FEATURES OF THE MODELS

2.1 *Monte Carlo simulation*

2.1.1 There are several features common to all the actuarial models. The first conspicuous one is that they are all designed to be used in Monte Carlo simulation exercises, and so in general can not be treated analytically, except in limited circumstances.

2.2 *Frequency*

2.2.1 The next common feature is that they are mostly defined in terms of *annual* steps, rather than more frequently. Such models are therefore not suitable for, and were not designed for, the pricing of derivatives. Exceptions are the Smith and Cairns models, which are defined as *continuous time* models, although they can be simulated with any desired frequency. Some of the actuarial models are consistent with a continuous time model, or at least partially so, or with simulation over any shorter intervals, such as monthly. We discuss this further in Section 4.5.

2.3 *Formulae*

2.3.1 Each model (except the Smith and Cairns models) is defined in terms of variables which take annual values, and are connected from one time step to the next by a series of formulae, very like recurrence relationships, but introducing for each basic variable at each time step some random *innovation* or *driver.* When analysing past data it is common to refer to these as random *residuals*, as indeed they are in those circumstances. But for the future they are the *innovations* that *drive* the process, and we prefer these terms. The Cairns model is defined in terms of stochastic differential equations, with driving Brownian motions instead of discrete innovations; The Smith model is defined in terms of continuous time equations, with driving compound Poisson diffusion processes. However, for both of these models, when they are simulated by Monte Carlo methods it is necessary to introduce random innovations for them too.

2.4 *Variables*

2.4.1 The first important feature that distinguishes the models when one wishes to use them is the set of output *variables* that they produce. Many of the variables come in necessary pairs, such as a price index and an annual rate of change of that index. Using the notation of the Wilkie model, we have:

The retail price index at time t is denoted by $Q(t)$.

The "force" (continuously compounded rate) of inflation over the year t–1 to t is denoted by $I(t)$, which is calculated, when analysing past data, as:

$I(t) = \ln Q(t) - \ln Q(t-1)$,

and for future simulation we have:

$Q(t) = Q(t–1).\exp(I(t))$.

We treat $Q(t)$ and $I(t)$, the retail prices index and the annual force of change in that index as *one* variable, rather than two. We could alternatively treat the force of inflation, $I(t)$, as a *basic variable*, since it is simulated directly, and the prices index, $Q(t)$, which is derived from $I(t)$, as a *derived variable*. However there are more complicated cases where we introduce derived variables, and it gives less clutter if we treat such pairs as one.

2.4.2 A more complicated case, however, is where share prices are *derived* from dividends and dividend yields. We use the Wilkie model again as an example: a dividend index, $D(t)$ is defined, and simulated (through its annual increments as the basic variable); a dividend yield, $Y(t)$ is also simulated. A share price index, $P(t)$, is derived from these by:

$P(t) = D(t)/Y(t)$.

A total return index for shares, $PR(t)$, can also be derived by:

$PR(t) = PR(t–1)\ [P(t) + D(t)(1–tax)]\ /\ P(t–1)$,

where *tax* is some chosen rate of tax, possibly zero, on dividends. Such a model gives us a value for dividend yield, and also allows us to treat dividend income differently from capital gains, so that we can allow, both for tax purposes and possibly for reinvestment expenses, for dividends received in cash. By contrast, a model that derives only a total return index on shares does not allow such detail; it may, however, be quite satisfactory for a simulation model that does not need to consider such details.

2.4.3 We show in Table 2.1 the variables that each of the models provides, distinguishing between series that are "base series" for the particular model, and those that are derived from base series. We also show the number of "drivers" used by each model, i.e. the number of separate innovations used for each year of each simulation, for the UK only. The number of drivers gives an indication of the complexity of the model. For the Smith model all the visible series are derived from a number of "hidden" series, that are stochastically generated from only four drivers.

2.4.4 All the models include price inflation, bonds, or interest rates on bonds, and shares, at least to some extent. The RWV and Smith models, however, only give total returns on shares, and not dividend yields. They also omit wages, though it would not be difficult to add them using Cairns's or Wilkie's approach to these. All, except Whitten & Thomas, include index-linked bonds. The Smith and Cairns models provide full yield curves, both for conventional and for index-linked bonds; none of the others do. Only the RWV, Wilkie and Smith models cover property (income and values in the case of the Wilkie model, and total return in the case of RWV and Smith). The TY model includes earnings on shares as well as dividends, and also covers overseas assets in sterling terms. The Wilkie model is the only one to cover exchange rates, though these could easily be incorporated into the other models in a similar way.

2.4.5 The Smith model is symmetric as between all asset classes, and may in theory be used to model "dividend yield" curves, analogously to those for fixed income and inflation-linked bonds, but the details of how to do this are not given by Smith (1996) nor by Huber (1998)). Again we would welcome details about the extension of the model in practice, although Smith & Speed (1998), discuss a hierarchy of models in general terms in a most illuminating way.

2.4.6 The Whitten & Thomas model is very similar in overall structure to the Wilkie model, and could easily be extended to cover index-linked, property and exchange rates by using the Wilkie model versions for these.

2.4.7 Although the Wilkie model provides only long-term par "consols" yields, $C(t)$, and a short-term interest "base rate", $B(t)$, an arbitrary par yield curve could be constructed for this model using the form:

$$\text{Par Yield}(t) = C(t) - (C(t) - B(t)).\exp(-a.t)$$

where a is chosen to be say 0.1 or 0.2, and is not stochastic. This produces plausible par yields at all terms. However, it does not guarantee that zero-coupon rates and forward rates remain non-negative at all terms unless the value of a is chosen carefully.

2.4.8 The Wilkie model explicitly models other countries on the same lines as the UK, and provides an exchange rate for each country modelled. In this respect it is wider than any of the other published models, though as mentioned above they too could presumably be applied to other countries, along with a model for exchange rates.

2.5 *Parameters*

2.5.1 Besides the formulae that define how each variable is simulated, each model requires certain *parameters*, but different sets of parameter values do not make a different model, only a different set of parameter values for that model. If the model has been calibrated from past data the authors have generally given the values of the parameters from their fitted model, but in general the models can be used with any parameter set that the user desires.

2.5.2 Ideally, in order to compare the models in certain respects we would have liked to have chosen parameter sets that gave similar medians for the variables under consideration. However, in the absence of "user friendly" formulae having been specified for the inputs, this is generally not a trivial task. Instead, for the purposes of this paper we have simply used the parameters as published.

2.5.3 In Table 2.1 we also show the number of parameters and initial conditions, required for each model, counting only the UK parts of each model. However, there is some uncertainty about these counts, for a variety of reasons (e.g whether Wilkie's DW and DX are counted as two parameters, or as one, since DW + DX normally = 1).

Table 2.1. Variables generated by the respective models

	Wilkie	Wilkie ARCH	RWV lognormal	RWV α-stable	Smith jump diffusion	TY	Cairns	Whitten & Thomas
Prices index	Base	Base	Base	Base	Derived	Base	Base	Base
Wages index	Base	Base				Base	Base	Base
Shares:								
Dividends	Base	Base				Derived	Derived	Base
Dividend yield	Base	Base				Base	Derived	Base
Earnings yield						Base		
Earnings						Base		
Price index	Derived	Derived				Derived	Base	Derived
Total return	Derived	Derived	Base	Base	Derived	Derived	Base	Derived
Interest rates								
"Consols" or long dated yield	Base	Base				Base	Derived	Base
Total return	Derived	Derived	Base	Base	Derived	Derived	Derived	Derived
Base rate	Base	Base			Derived	Base	Derived	Base
Total return	Derived	Derived	Base	Base	Derived	Derived	Derived	Derived
Yield curve	Derived	Derived			Derived		Base	Derived
Index-linked real Yields:								
I-L yield	Base	Base			Derived	Base	Derived	
Total return	Derived	Derived	Base	Base	Derived	Derived	Derived	
Short-term I-L Yield					Derived		Derived	
Total return					Derived		Derived	
Yield curve					Derived		Base	

Table 2.1 (continued). Variables generated by the respective models

	Wilkie	Wilkie ARCH	RWV lognormal	RWV α-stable	Smith jump diffusion	TY	Cairns	Whitten & Thomas
Property:								
Income	Base	Base						
Income yield	Base	Base						
Price index	Derived	Derived						
Total return	Derived	Derived			Derived			
Number of drivers for the UK model	9	9	6	6	4	8	6	6
Number of UK variables modelled	9	9	6	6	5 + nominal and index-linked yield curves	8	3 + nominal and index-linked yield curves	6
Number of parameters	47	50	13 plus a 6 by 6 correlation matrix	15 plus a 6 by 6 correlation matrix	41	45	38	57
Number of initial conditions	17	17	1 (inflation)	1 (inflation)	Initial yield curves for each series	17	6	12
Exchange rate	Base	Base						
Overseas investments								
Total return	Derived	Derived				Base		

2.6 *Range of the variables*

2.6.1 Within each model each variable has a certain range of values that it may attain. A typical range for an index type of variable is 0 to $+\infty$, and for the annual force of change of that variable is $-\infty$ to $+\infty$. It is most unlikely that such a variable will even approach either of these extremes, but no value within the ranges is inherently impossible, so a simulated variable that allows such a range is acceptable. We make no comment on such variables.

2.6.2 The situation is different for interest rates. It is in principle unacceptable for the value of a variable that represents nominal interest rates to be negative. A model that allows such a possibility may need modification to exclude it. The Wilkie model would allow the nominal "consols" yield, $C(t)$, to become negative if inflation were sufficiently negative and the "real yield" part were not large enough to counter this. In practical application of the Wilkie model one inserts a minimum value for $C(t)$, denoted $CMIN$, as one of the parameters, with a typical value of 0.5%. This value is used in place of the simulated value in any year that it applies. Such a barrier may either be applied without affecting future simulated values, or it may affect the carried forward value of the relevant variable as used in the simulation for next year. If such minima need to be applied it is not always clear what authors of models would recommend. We found in our simulation work that a similar minimum is needed for the TY model.

2.6.3 In the case of real interest rates, such as might apply to index-linked bonds, it is less obvious that negative yields are impossible in practice. An index-linked zero-coupon bond might well sell "above par", and hence at a negative real yield, if circumstances were thought to favour this. However, it is implausible that an index-linked bond with a negative coupon would be issued. The practical difficulty for the borrower in collecting the negative interest payments would not worry a simulation exercise, but the market value of such a bond could become negative if real yields rose to be sufficiently positive at some later point of the simulation, and this could be undesirable. It might therefore be desirable to restrict investment in index-linked bonds either to zero-coupons at all times, or to zero-coupons bonds if the simulated real yields were negative. This does not involve a discontinuity in procedure: as real yields move down to zero the coupon approaches zero, and at zero a coupon bond and a zero-coupon bond are the same; but it does imply an asymmetry of treatment when real bond yields are on either side of zero.

2.7 *Innovations*

2.7.1 Each model is driven by a series of innovations. These are always treated as independent from year to year and from simulation to simulation. If they were to be treated as not independent then the dependence should enter the "skeleton" of the model, that is, the deterministic part before the innovations are added. In many of the models the innovations are modelled as a series of independent identically distributed normal variates. However, Wilkie's ARCH model allows for a varying standard deviation in the inflation model, and Whitten & Thomas allow for different standard deviations in the two states of their model. Their innovations are still normally distributed. However, Smith's innovations in effect consist of the differences between two sets of independent gamma-distributed variables and the RWV α-stable model uses independent α-stable variates (also known as Lévy-stable or stable Paretian variates).

2.7.2 We discuss alternative ways in which innovations could be modelled in Section 4.4.

2.8 *Initial conditions*

2.8.1 Every stochastic economic model needs some initial conditions, that is values of the state space at time $t = 0$. However, in some cases the initial conditions may be arbitrary and may have no effect on the effective development of the model.

2.8.2 For example, in a random walk model:

$$X(t) = X(t{-}1).\exp(XMU + XE(t)),$$

where XMU is constant and $XE(t)$ is some random innovation, then the value of $X(0)$ is just a scale factor for the index, $X(.)$, and does not affect any of the returns that one might calculate over any intervals. In such cases the value of $X(0)$ could be set arbitrarily, say at 1 or 100. It might be convenient to be able to insert some other value, for example the current value of the Retail Prices Index or the FTSE Actuaries All-share Index, but this is a convenience, rather than a necessity, and we shall ignore such initial conditions in our discussion.

2.8.3 On the other hand, for an AR(1) model, say:

$$X(t) = XMU + XA.(X(t{-}1) - XMU) + XE(t),$$

then the value of $X(0)$ has a significant effect on the development of the simulations. The experience when the value of $X(0)$ is high will be different from what it would be when $X(0)$ is low. We note such cases as effective initial conditions for each of the models.

2.8.4 In many cases we can define what we call neutral initial conditions. In the example above $X(0) = XMU$ would be neutral. These can be defined in a number of ways. One is as what the model would settle down to asymptotically if all future innovations were zero; or equivalently, the set of initial conditions such that $X(t) = X(0)$ for all future t if all the innovations were to be zero. Typically they are the (unconditional) *median* values of $X(t)$ (unconditional in the sense that they are the median values of the distribution of $X(t)$ as t goes to infinity, so that the effect of the initial conditions has worn off).

2.8.5 In the example above the neutral initial conditions as just defined would also be the mean value of $X(\mathrm{t})$. But in the model:

$$\ln X(t) = \ln XMU + XA(\ln X(t-1) - XMU) + XSD.XZ(t),$$

where $XZ(t)$ is i.i.d. unit normal, whilst the unconditional median of $X(t)$ is XMU, the unconditional mean of $X(t)$ is $XMU.\exp(\tfrac{1}{2}XSD^2)$. It would be possible therefore to define neutral initial conditions instead as the unconditional *means* of the various state variables in each model, taking account of the structure of the model and distribution of the residuals, but this may well be rather harder to calculate then the way we have defined them.

2.8.6 In some models there may be no unique neutral initial condition. Consider a model where the system may be in one of two or more states, with a system of transition probabilities determining how moves between states happen. Each simulation must start in some specific state, but no state is neutral in our sense. Instead it may be best to allow the initial state to be selected at random for each simulation, with specified probabilities. If the chosen probabilities are those that are the long-term probabilities of the states themselves, then one can speak of a neutral *probability distribution* for the initial state. But one might well wish to simulate conditional on knowing what the initial state is, so one may wish to use some non-neutral probability structure.

2.8.7 One can take this idea further, and start each simulation in a random state such that the long-run probability distribution of the state variables is replicated. It may be complicated to calculate the joint probability distribution of all the state variables analytically and then simulate from this joint distribution, and an alternative way of reproducing this idea might be to run each simulation for say 100 years from some fixed starting state and then start with the simulated position at time $t = 100$, recording only the changes from that point onwards. However, this method relies on the model being ergodic, i.e. (roughly) reaching some steady state distribution in due course. Most of the models are ergodic, though Smith's jump diffusion model appears not to be in the long run, as noted by Huber.

2.8.8 In Whitten & Thomas's model it is not obvious what either the median or mean values of the variables are. To calculate them analytically seems complicated, and the distribution is not an obvious one. There are two sets of values that could be thought of as neutral, one for each state, in the sense that, if started with these values, and with zero innovations, the values of the variables would remain the same. We have started the simulations using these neutral values for the lower state.

2.8.9 To derive initial conditions suitable for simulation from a set of conditions observable in the market may not be trivially easy. Neither Smith nor Cairns explain in their published papers how this can be done for their models, and we have not considered it in detail. It is a matter of finding a set of state variables within the simulation model which reproduce a given yield curve in some way. It may not be possible to do this exactly, and there is then a choice as to which error function one wishes to adopt. It would be made easier if an unambiguous zero coupon yield curve for the market were immediately available, but the construction of such a yield curve remains a matter of choice at present.

2.9 *Different currencies*

2.9.1 Where the models include foreign (overseas) investments (and some do not) three different approaches have been adopted.

2.9.2 Wilkie models investments in each currency separately, though with allowance for simultaneous correlation in the contemporaneous innovations across currencies, and then links these together through models for exchange rates.

2.9.3 Another approach, used in the TY model (and to our knowledge used by others via random walk extensions to a single country Wilkie model), is to model overseas investments directly in sterling.

2.9.4 Smith adopts a third approach, which is to model each asset class (including sterling) in an unobservable notional risk neutral currency, and then to derive sterling values of assets by taking ratios of the notional currency values of those assets to the notional currency value of sterling. This in theory would permit the published Smith model to be extended to include, for example, both a US dollar and a US equities series, thus permitting (depending on the denominator used in the ratio) both sterling and US dollar values of US equities to be calculated. However, details of whether such extensions are actually feasible in practice (including the difficult issue of calibrating the necessary additional parameters) have to the best of our knowledge not yet been published.

2.9.5 It would not be too difficult within each of the other models to follow the Wilkie approach, to model domestic investments in each currency separately, and then to use the Wilkie, or some other, model, for exchange rates, but this does not seem to have been implemented by anyone else.

3 RESULTS

3.1 *Results*

3.1.1 In Tables 3.1a to 3.7b below we show numerical results from the different models on the same lines as shown in Tables 11.1 and 11.2 of Wilkie (1995). We follow the notation of Wilkie (1995). Consider any variable, $X(t)$, such as a price index or a total return index. We then define nominal returns as:

$FX(t) = X(t)/X(0)$

$GX(t) = 100\{FX(t)^{1/t} - 1\}$

and real returns (relative to price inflation) as:

$HX(t) = FX(t)/FQ(t)$

$JX(t) = 100\{HX(t)^{1/t} - 1\}$

3.1.2 We then denote the various series using Wilkie's general notation:

- Q: retail price index
- W wages index
- PR share total return index
- CR long-term bond total return index
- BR "cash" or short-term bond total return index
- RR index linked (long bond) total return index
- AR property total return index.

3.1.3 How these total return indices are calculated varies from model to model. There is no divergence for the retail price index and the wages index; the index value is used.

3.1.4 For ordinary shares there are three approaches: where the total return index is simulated (RWV and Smith) that is used. Where there are separate price and dividend indices, the dividends can be compounded annually (Wilkie, TY, Whitten & Thomas). Cairns models the total return index directly and derives the share price index, dividend index and dividend yield from that. Property is treated by Wilkie and Smith in the same way as they treat shares.

3.1.5 For bonds there are also different approaches, and some models allow different possibilities to be calculated. The RWV models model total returns directly, for long-term bonds of unspecified term. Wilkie (as also Whitten & Thomas) assumes that his long-term bond yield, $C(t)$, is the yield on an irredeemable stock with interest paid annually, and compounds the returns annually. For index-linked bonds Wilkie assumes investment in a 15-year par bond which the following year has become a 14-year bond; it is then sold and reinvested in a new 15-year par bond, all transactions taking place at the same yield, $R(t)$. For the Cairns model, where full yield curves are available we have followed this approach, but using as a long-term conventional bond a 30-year par bond that reduces to 29 years and is then reinvested, in both cases valuing the bonds using the full zero-coupon curve; for index-linked in the Cairns model we have done as Wilkie, with a 15-year bond reducing to 14 years, but with prices determined by the full set of zero-coupon rates. Smith's total returns on bonds are calculated assuming continuous rolling reinvestment into a zero-coupon bond with a constant duration; we have assumed a 30-year bond as the long-term conventional, and a 15-year bond for index-linked.

3.1.6 For short-term bonds there are also different approaches. The RWV model again models the total return directly. Smith also models the total return, assuming that a cash deposit earns interest continuously at the spot rate. Wilkie (as also Whitten & Thomas) assumes that his short-term bond yield, $B(t)$, is a one-year rate, compounded annually. We use Cairns's one-year rates, also compounded annually.

3.1.7 Although we have calculated values for the RWV models with α-stable innovations, we do not show results. The means and standard deviations for α-stable distributions are theoretically infinite. Only the simulated medians have sense; simulated means and standard deviations are of course not infinite, but are extremely large and very variable. It would be better to have calculated some inter-percentile range that could be compared with the standard deviations from the other models, but we have not had time to do this.

3.1.8 For each model we have used neutral initial conditions as far as we can. For Whitten & Thomas we have used the neutral initial conditions for state 1. An alternative approach would have been to start each model from the same set of market conditions, though the results might have been very dependent on which set one chose. It is not a trivial job to derive the equivalent state space for simulation from what is available to describe market conditions. It is not even always trivial to calculate neutral initial conditions (even in our stationary sense), and none of our authors has described fully what to do (where it matters).

3.1.9 In Table (a) for each model (except the RWV α-stable) we show values measured in nominal terms, and in Table (b) values measured in real terms (obviously the real return on price inflation is zero, so it is omitted). Wilkie (1995) showed means, standard deviations and correlation coefficients, based on 1,000 simulations. We show medians instead of means, and have used 10,000 simulations. The advantage of medians is that, since many of the distributions are close to lognormal, the median corresponds to "$\exp(\mu)$", whereas the mean corresponds to "$\exp(\mu+½\sigma^2)$" and is therefore distorted when the standard deviation is relatively large.

3.1.10 In the Tables $M(X)$ is the median and $SD(X)$ the standard deviation of X; and $C(X,Y)$ is the correlation coefficient between X and Y.

3.2 *Discussion*

3.2.1 When looking at the results displayed in Tables 3.1a to 3.7b, one can consider three elements: the medians, standard deviations, and correlation coefficients. As noted in section 2.5 it should be possible to alter the parameters of the models to adjust the medians (or means) in the way that one wishes, though it is not always immediately obvious with the more complex models how this is done. Especially one may be concerned with the relative means, i.e. the difference between the nominal means of the returns on any two series. In general the medians are similar for any one series and any one variable for all durations, so a change to one parameter is likely to point the whole distribution in the same way.

3.2.2 The parameters that control the standard deviations in the models generally act on the one year standard deviations, and the way in which those standard deviations behave with duration is a function of the structure of the skeleton of the model. One can see, for example, how the standard deviations for GQ in the Wilkie model are initially large and decrease with duration, whereas for the Wilkie ARCH model they start at a lower level, but remain at much the same level throughout. The Smith and Cairns models, on the other hand, have very low values of SD(GQ) over one year, rising to much higher values than Wilkie by 40 years.

3.2.3 The way in which the correlation coefficients develop with duration also depends on the structure of the skeleton of the model. For example, in the Wilkie model, C(GCR, GQ), the correlation coefficient between the returns on long bonds an inflation, is –0.32 over one year, becoming more negative to –0.54 over 10 years, and changing to positive 0.37 by 40 years. In the Smith model the corresponding coefficients are negative all the way up to 40 years, though changing uniformly from –0.54 at one year to –0.11 at 40 years.

Table 3.1a. Wilkie model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	4.75	4.83	4.82	4.84	4.81	4.84
SD(GQ)	4.50	4.18	3.66	2.99	2.26	1.64
Mean rate of growth of nominal wages, GW						
M(GW)	6.34	6.41	6.41	6.44	6.40	6.42
SD(GW)	3.74	3.57	3.24	2.69	2.05	1.50
C(GW,GQ)	0.74	0.86	0.94	0.96	0.96	0.96
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	10.89	10.79	10.82	10.94	10.91	10.97
SD(GPR)	19.34	12.95	7.21	4.99	3.62	2.62
C(GPR,GQ)	–0.29	–0.13	0.19	0.41	0.55	0.62
C(GPR,GW)	–0.21	–0.08	0.20	0.40	0.53	0.60
Mean rate of nominal total return on long bonds, GCR						
M(GCR)	7.75	7.65	7.64	7.73	7.91	7.98
SD(GCR)	7.84	5.40	2.94	1.64	1.04	1.05
C(GCR,GQ)	–0.32	–0.42	–0.55	–0.54	–0.15	0.37
C(GCR,GW)	–0.24	–0.36	–0.51	–0.51	–0.13	0.36
C(GCR,GPR)	0.32	0.26	0.06	–0.07	0.05	0.30
Mean rate of nominal total return on cash, GBR						
M(GBR)	6.16	6.17	6.22	6.30	6.41	6.51
SD(GBR)	0.00	0.63	1.11	1.30	1.33	1.23
C(GBR,GQ)	0.00	0.12	0.22	0.32	0.44	0.56
C(GBR,GW)	0.00	0.12	0.21	0.31	0.42	0.54
C(GBR,GPR)	0.00	–0.05	0.04	0.13	0.25	0.36
C(GBR,GCR)	0.00	–0.25	–0.32	–0.25	0.25	0.69
Mean rate of nominal total return on index-linked bonds, GRR						
M(GRR)	8.93	8.99	9.00	9.04	8.99	9.02
SD(GRR)	5.46	4.59	3.85	3.13	2.36	1.70
C(GRR,GQ)	0.86	0.94	0.99	0.99	1.00	1.00
C(GRR,GW)	0.63	0.81	0.92	0.95	0.96	0.96
C(GRR,GPR)	–0.26	–0.12	0.19	0.41	0.54	0.62
C(GRR,GCR)	0.02	–0.21	–0.48	–0.49	–0.12	0.40
C(GRR,GBR)	0.00	0.08	0.21	0.33	0.45	0.57
Mean rate of nominal total return on property, GAR						
M(GAR)	12.70	12.91	13.06	13.13	13.13	13.16
SD(GAR)	11.65	7.79	4.32	3.12	2.79	2.46
C(GAR,GQ)	0.06	0.12	0.29	0.51	0.62	0.61
C(GAR,GW)	0.06	0.12	0.27	0.50	0.59	0.58
C(GAR,GPR)	–0.02	–0.01	0.07	0.25	0.37	0.40
C(GAR,GCR)	–0.01	–0.04	–0.15	–0.23	0.01	0.31
C(GAR,GBR)	0.00	0.02	0.07	0.20	0.32	0.38
C(GAR,GRR)	0.05	0.11	0.28	0.51	0.61	0.60

Table 3.1b Wilkie model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of real growth of wages, JW						
M(JW)	1.50	1.49	1.49	1.49	1.50	1.50
SD(JW)	2.94	2.05	1.27	0.88	0.62	0.43
C(JW,GQ)	–0.58	–0.54	–0.51	–0.51	–0.50	–0.48
Mean rate of growth of real total return on shares, JPR						
M(JPR)	5.86	5.71	5.66	5.79	5.89	5.84
SD(JPR)	20.41	13.58	7.17	4.47	2.92	1.97
C(JPR,GQ)	–0.49	–0.43	–0.34	–0.24	–0.14	–0.05
C(JPR,JW)	0.30	0.28	0.22	0.16	0.09	0.05
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	2.51	2.51	2.55	2.68	2.87	2.98
SD(JCR)	9.84	7.82	5.65	4.01	2.57	1.55
C(JCR,GQ)	–0.69	–0.80	–0.91	–0.94	–0.92	–0.80
C(JCR,JW)	0.40	0.43	0.47	0.48	0.47	0.39
C(JCR,JPR)	0.50	0.46	0.38	0.29	0.19	0.09
Mean rate of growth of real total return on cash, JBR						
M(JBR)	1.34	1.32	1.42	1.52	1.63	1.64
SD(JBR)	4.35	4.01	3.46	2.75	1.99	1.35
C(JBR,GQ)	–1.00	–0.99	–0.95	–0.90	–0.82	–0.69
C(JBR,JW)	0.58	0.54	0.49	0.46	0.41	0.33
C(JBR,JPR)	0.49	0.42	0.32	0.22	0.12	0.05
C(JBR,JCR)	0.69	0.77	0.84	0.84	0.83	0.82
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	4.04	4.01	3.98	3.99	3.99	3.99
SD(JRR)	2.68	1.49	0.60	0.33	0.19	0.11
C(JRR,GQ)	0.00	–0.02	–0.01	0.00	0.01	–0.01
C(JRR,JW)	–0.01	0.00	0.00	0.00	0.01	–0.01
C(JRR,JPR)	–0.01	0.01	0.00	0.00	0.00	0.01
C(JRR,JCR)	0.44	0.37	0.20	0.13	0.14	0.24
C(JRR,JBR)	0.00	0.01	0.00	0.04	0.12	0.21
Mean rate of growth of real total return on property, JAR						
M(JAR)	7.57	7.71	7.86	7.85	7.91	7.93
SD(JAR)	11.82	8.15	4.72	2.99	2.21	1.89
C(JAR,GQ)	–0.34	–0.42	–0.55	–0.52	–0.32	–0.14
C(JAR,JW)	0.21	0.25	0.29	0.28	0.16	0.06
C(JAR,JPR)	0.16	0.19	0.20	0.17	0.10	0.04
C(JAR,JCR)	0.24	0.34	0.50	0.50	0.34	0.18
C(JAR,JBR)	0.34	0.42	0.52	0.48	0.30	0.14
C(JAR,JRR)	0.00	0.01	0.00	0.00	0.00	–0.01

Table 3.2a. Wilkie ARCH model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	4.04	4.10	4.10	4.11	4.07	4.09
SD(GQ)	2.69	2.73	3.00	3.30	3.04	2.47
Mean rate of growth of nominal wages, GW						
M(GW)	5.74	5.75	5.77	5.78	5.75	5.78
SD(GW)	3.00	2.65	2.67	2.90	2.69	2.19
C(GW,GQ)	0.56	0.74	0.91	0.96	0.98	0.98
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	10.03	9.99	10.02	10.10	10.12	10.13
SD(GPR)	18.62	12.70	7.04	5.01	4.18	3.52
C(GPR,GQ)	–0.18	–0.11	0.09	0.38	0.66	0.78
C(GPR,GW)	–0.09	–0.06	0.11	0.38	0.65	0.77
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	7.12	7.01	6.96	7.04	7.21	7.30
SD(GCR)	8.32	5.64	3.15	2.36	1.42	1.25
C(GCR,GQ)	–0.20	–0.30	–0.49	–0.60	–0.34	0.26
C(GCR,GW)	–0.12	–0.23	–0.45	–0.58	–0.32	0.26
C(GCR,GPR)	0.27	0.25	0.12	–0.10	–0.09	0.30
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	5.60	5.61	5.66	5.74	5.84	5.94
SD(GBR)	0.00	0.58	1.02	1.21	1.31	1.33
C(GBR,GQ)	0.00	0.08	0.16	0.28	0.47	0.63
C(GBR,GW)	0.00	0.07	0.15	0.27	0.46	0.62
C(GBR,GPR)	0.00	–0.06	0.01	0.11	0.32	0.51
C(GBR,GCR)	0.00	–0.25	–0.32	–0.26	0.07	0.61
Mean rate of growth of nominal total return on index-linked bonds, GRR						
M(GRR)	8.19	8.24	8.23	8.27	8.21	8.25
SD(GRR)	3.95	3.21	3.17	3.45	3.17	2.57
C(GRR,GQ)	0.71	0.87	0.98	1.00	1.00	1.00
C(GRR,GW)	0.39	0.64	0.89	0.96	0.98	0.98
C(GRR,GPR)	–0.13	–0.10	0.09	0.38	0.66	0.78
C(GRR,GCR)	0.28	0.01	–0.40	–0.57	–0.32	0.27
C(GRR,GBR)	0.00	0.02	0.15	0.29	0.48	0.64
Mean rate of growth of nominal total return on property, GAR						
M(GAR)	11.94	12.12	12.25	12.34	12.35	12.35
SD(GAR)	11.56	7.71	4.20	3.04	3.07	2.94
C(GAR,GQ)	0.04	0.08	0.22	0.48	0.69	0.74
C(GAR,GW)	0.04	0.08	0.20	0.47	0.68	0.73
C(GAR,GPR)	–0.01	0.00	0.03	0.23	0.50	0.60
C(GAR,GCR)	0.00	–0.02	–0.10	–0.28	–0.14	0.29
C(GAR,GBR)	0.00	0.01	0.04	0.17	0.38	0.52
C(GAR,GRR)	0.02	0.07	0.21	0.48	0.69	0.74

Table 3.2b Wilkie ARCH model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of real growth of wages, JW						
M(JW)	1.60	1.59	1.59	1.58	1.60	1.60
SD(JW)	2.60	1.88	1.24	0.92	0.69	0.49
C(JW,GQ)	–0.39	–0.42	–0.48	–0.55	–0.61	–0.63
Mean rate of growth of real total return on shares, JPR						
M(JPR)	5.70	5.60	5.67	5.70	5.84	5.79
SD(JPR)	18.62	12.82	7.15	4.68	3.04	2.05
C(JPR,GQ)	–0.32	–0.32	–0.33	–0.31	–0.17	0.03
C(JPR,JW)	0.14	0.17	0.21	0.23	0.16	0.02
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	2.73	2.68	2.61	2.73	2.95	3.03
SD(JCR)	8.92	6.73	5.20	5.10	3.79	2.42
C(JCR,GQ)	–0.47	–0.64	–0.85	–0.89	–0.90	–0.84
C(JCR,JW)	0.18	0.26	0.41	0.52	0.58	0.56
C(JCR,JPR)	0.36	0.37	0.38	0.35	0.24	0.06
Mean rate of growth of real total return on cash, JBR						
M(JBR)	1.50	1.49	1.60	1.68	1.79	1.82
SD(JBR)	2.62	2.67	2.91	3.01	2.58	1.82
C(JBR,GQ)	–1.00	–0.98	–0.94	–0.92	–0.88	–0.83
C(JBR,JW)	0.39	0.41	0.46	0.52	0.56	0.53
C(JBR,JPR)	0.32	0.30	0.32	0.30	0.18	0.01
C(JBR,JCR)	0.47	0.59	0.76	0.85	0.89	0.89
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	4.04	4.01	3.98	3.99	3.99	3.99
SD(JRR)	2.68	1.49	0.60	0.33	0.19	0.11
C(JRR,GQ)	0.00	–0.02	–0.01	0.00	0.01	–0.01
C(JRR,JW)	–0.01	0.00	0.01	0.00	0.01	0.00
C(JRR,JPR)	–0.01	0.00	0.01	0.00	0.00	0.01
C(JRR,JCR)	0.53	0.46	0.24	0.12	0.09	0.16
C(JRR,JBR)	0.00	0.00	0.00	0.04	0.09	0.16
Mean rate of growth of real total return on property, JAR						
M(JAR)	7.55	7.73	7.83	7.83	7.91	7.93
SD(JAR)	11.37	7.72	4.52	3.19	2.38	1.94
C(JAR,GQ)	–0.21	–0.29	–0.49	–0.60	–0.44	–0.22
C(JAR,JW)	0.10	0.14	0.24	0.35	0.30	0.13
C(JAR,JPR)	0.06	0.10	0.18	0.23	0.15	0.05
C(JAR,JCR)	0.11	0.19	0.42	0.56	0.46	0.25
C(JAR,JBR)	0.21	0.28	0.46	0.57	0.44	0.22
C(JAR,JRR)	0.00	0.00	0.00	0.00	0.00	–0.01

Table 3.3a. RW normal model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	4.73	4.84	4.82	4.85	4.81	4.85
SD(GQ)	5.94	5.52	4.83	3.95	2.98	2.16
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	16.68	16.45	16.41	16.43	16.28	16.23
SD(GPR)	26.28	18.19	11.84	8.54	6.07	4.34
C(GPR,GQ)	0.00	0.08	0.23	0.29	0.32	0.34
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	10.25	9.91	9.77	9.74	9.69	9.66
SD(GCR)	14.00	9.88	6.44	4.66	3.39	2.44
C(GCR,GQ)	–0.13	0.02	0.25	0.36	0.41	0.42
C(GCR,GPR)	0.54	0.54	0.58	0.60	0.61	0.62
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	6.40	6.42	6.41	6.43	6.34	6.37
SD(GBR)	3.96	3.71	3.46	2.91	2.21	1.62
C(GBR,GQ)	0.42	0.70	0.87	0.91	0.93	0.93
C(GBR,GPR)	0.04	0.10	0.23	0.29	0.32	0.34
C(GBR,GCR)	0.09	0.15	0.31	0.40	0.44	0.45
Mean rate of growth of nominal total return on index-linked bonds, GRR						
M(GRR)	9.25	9.32	9.22	9.21	9.18	9.11
SD(GRR)	14.68	10.85	7.52	5.60	4.11	2.96
C(GRR,GQ)	0.29	0.41	0.57	0.63	0.66	0.66
C(GRR,GPR)	0.46	0.46	0.52	0.55	0.56	0.57
C(GRR,GCR)	0.81	0.82	0.85	0.87	0.88	0.89
C(GRR,GBR)	0.24	0.38	0.55	0.62	0.65	0.66
Mean rate of growth of nominal total return on property, GAR						
M(GAR)	10.72	10.86	10.93	11.06	11.03	10.96
SD(GAR)	27.41	19.18	12.14	8.66	6.21	4.44
C(GAR,GQ)	0.04	0.11	0.22	0.28	0.31	0.34
C(GAR,GPR)	0.30	0.30	0.34	0.37	0.39	0.39
C(GAR,GCR)	–0.06	–0.07	0.00	0.05	0.08	0.08
C(GAR,GBR)	–0.24	–0.10	0.07	0.15	0.20	0.22
C(GAR,GRR)	–0.01	0.01	0.10	0.16	0.19	0.20

Table 3.3b RW normal model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of growth of real total return on shares, JPR						
M(JPR)	11.47	11.03	10.99	10.97	10.94	10.94
SD(JPR)	25.96	17.90	11.26	8.01	5.63	3.99
C(JPR,GQ)	–0.25	–0.25	–0.22	–0.23	–0.23	–0.22
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	5.09	4.84	4.55	4.63	4.65	4.62
SD(JCR)	15.37	10.84	6.78	4.75	3.39	2.42
C(JCR,GQ)	–0.50	–0.49	–0.48	–0.49	–0.49	–0.49
C(JCR,JPR)	0.58	0.58	0.58	0.59	0.58	0.58
Mean rate of growth of real total return on cash, JBR						
M(JBR)	1.50	1.44	1.49	1.44	1.45	1.46
SD(JBR)	5.40	3.80	2.40	1.70	1.21	0.85
C(JBR,GQ)	–0.77	–0.75	–0.74	–0.75	–0.76	–0.76
C(JBR,JPR)	0.22	0.22	0.20	0.21	0.21	0.21
C(JBR,JCR)	0.48	0.48	0.47	0.48	0.47	0.47
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	4.40	4.28	4.20	4.13	4.09	4.11
SD(JRR)	13.48	9.50	5.95	4.17	2.96	2.13
C(JRR,GQ)	–0.14	–0.13	–0.12	–0.13	–0.13	–0.13
C(JRR,JPR)	0.49	0.48	0.50	0.50	0.50	0.50
C(JRR,JCR)	0.84	0.84	0.83	0.83	0.83	0.84
C(JRR,JBR)	0.19	0.19	0.19	0.19	0.19	0.19
Mean rate of growth of real total return on property, JAR						
M(JAR)	5.83	5.77	5.75	5.86	5.88	5.87
SD(JAR)	26.70	18.56	11.53	8.11	5.75	4.06
C(JAR,GQ)	–0.20	–0.20	–0.20	–0.21	–0.20	–0.19
C(JAR,JW)	0.05	0.05	0.05	0.05	0.05	0.06
C(JAR,JPR)	0.34	0.33	0.33	0.35	0.35	0.34
C(JAR,JCR)	0.05	0.04	0.04	0.05	0.06	0.03
C(JAR,JBR)	–0.02	–0.02	–0.02	–0.01	–0.01	–0.03
C(JAR,JRR)	0.00	–0.01	–0.01	0.00	0.01	–0.01

Table 3.4a. Smith model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	3.96	3.93	3.85	3.72	3.49	3.05
SD(GQ)	0.38	0.59	0.95	1.35	1.90	2.68
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	16.82	15.51	14.60	14.00	13.28	12.58
SD(GPR)	21.33	15.29	9.41	6.52	4.52	3.66
C(GPR,GQ)	–0.18	–0.16	–0.10	–0.04	0.09	0.31
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	13.79	12.30	10.47	9.81	9.05	8.40
SD(GCR)	23.17	16.49	9.98	6.45	3.77	1.92
C(GCR,GQ)	–0.54	–0.48	–0.45	–0.42	–0.36	–0.11
C(GCR,GPR)	0.44	0.43	0.36	0.27	0.10	0.22
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	7.79	7.65	7.30	6.94	6.30	5.71
SD(GBR)	0.40	0.28	0.74	1.24	1.90	2.80
C(GBR,GQ)	–0.54	0.24	0.52	0.53	0.53	0.52
C(GBR,GPR)	0.43	0.02	–0.23	–0.15	0.09	0.48
C(GBR,GCR)	0.99	0.01	–0.69	–0.74	–0.66	–0.23
Mean rate of growth of nominal total return on index-linked bonds, GRR						
M(GRR)	9.76	9.32	8.70	8.17	7.51	6.84
SD(GRR)	11.12	7.80	4.75	3.25	2.47	2.63
C(GRR,GQ)	0.47	0.44	0.53	0.68	0.85	0.85
C(GRR,GPR)	0.27	0.26	0.20	0.15	0.15	0.48
C(GRR,GCR)	0.48	0.47	0.38	0.22	–0.06	–0.02
C(GRR,GBR)	0.48	0.03	–0.19	–0.02	0.38	0.79
Mean rate of growth of nominal total return on property, GAR						
M(GAR)	11.50	10.49	9.53	9.01	8.27	7.61
SD(GAR)	15.53	11.15	7.10	5.09	4.03	3.82
C(GAR,GQ)	0.08	0.07	0.13	0.19	0.30	0.51
C(GAR,GPR)	0.28	0.27	0.25	0.24	0.30	0.58
C(GAR,GCR)	0.04	0.01	–0.07	–0.18	–0.32	–0.12
C(GAR,GBR)	0.03	0.02	0.10	0.24	0.48	0.75
C(GAR,GRR)	0.12	0.09	0.07	0.07	0.25	0.68

Table 3.4b Smith model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of growth of real total return on shares, JPR						
M(JPR)	12.31	11.15	10.40	9.91	9.52	9.25
SD(JPR)	20.60	14.82	9.23	6.50	4.64	3.82
C(JPR,GQ)	–0.20	–0.20	–0.21	–0.26	–0.35	–0.46
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	9.51	8.13	6.42	5.98	5.43	5.21
SD(JCR)	22.49	16.16	10.08	6.91	4.71	3.48
C(JCR,GQ)	–0.55	–0.51	–0.52	–0.58	–0.69	–0.85
C(JCR,JPR)	0.45	0.45	0.40	0.37	0.34	0.52
Mean rate of growth of real total return on cash, JBR						
M(JBR)	3.71	3.57	3.38	3.14	2.81	2.61
SD(JBR)	0.67	0.59	0.84	1.24	1.82	2.63
C(JBR,GQ)	–0.87	–0.89	–0.69	–0.57	–0.51	–0.48
C(JBR,JPR)	0.36	0.20	0.00	0.03	0.22	0.53
C(JBR,JCR)	0.88	0.51	–0.01	–0.11	–0.01	0.31
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	5.60	5.18	4.70	4.32	3.92	3.69
SD(JRR)	10.54	7.27	4.14	2.44	1.27	1.43
C(JRR,GQ)	0.44	0.38	0.36	0.32	0.09	–0.36
C(JRR,JPR)	0.27	0.27	0.20	0.13	0.09	0.53
C(JRR,JCR)	0.49	0.50	0.46	0.40	0.30	0.38
C(JRR,JBR)	0.04	–0.37	–0.69	–0.65	–0.17	0.81
Mean rate of growth of real total return on property, JAR						
M(JAR)	7.30	6.33	5.49	5.10	4.64	4.48
C(JAR,GQ)	0.06	0.02	–0.02	–0.09	–0.20	–0.25
C(JAR,JPR)	0.29	0.27	0.26	0.26	0.33	0.56
C(JAR,JCR)	0.05	0.04	–0.01	–0.04	–0.03	0.18
C(JAR,JBR)	–0.01	–0.01	0.04	0.18	0.44	0.68
C(JAR,JRR)	0.11	0.07	0.00	–0.10	–0.03	0.57

Table 3.5a. TY model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	3.96	4.02	4.04	4.06	4.01	4.03
SD(GQ)	3.22	3.32	3.58	3.55	2.91	2.19
Mean rate of growth of nominal wages, GW						
M(GW)	6.03	6.04	6.03	6.09	6.02	6.03
SD(GW)	3.00	2.85	3.00	2.98	2.48	1.88
C(GW,GQ)	0.58	0.70	0.86	0.92	0.94	0.95
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	9.05	9.23	9.04	9.10	9.22	9.38
SD(GPR)	16.38	9.92	5.67	3.84	2.77	2.11
C(GPR,GQ)	–0.01	–0.01	0.07	0.23	0.50	0.67
C(GPR,GW)	–0.01	0.00	0.10	0.26	0.54	0.70
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	6.77	6.97	6.97	7.04	7.14	7.09
SD(GCR)	17.02	10.45	5.93	3.61	1.86	1.14
C(GCR,GQ)	–0.31	–0.43	–0.58	–0.47	0.00	0.35
C(GCR,GW)	–0.17	–0.30	–0.48	–0.41	0.03	0.36
C(GCR,GPR)	0.00	0.03	–0.02	–0.03	0.14	0.37
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	5.00	5.00	5.05	5.05	5.02	5.01
SD(GBR)	0.00	0.95	1.52	1.64	1.46	1.15
C(GBR,GQ)	0.00	0.13	0.31	0.45	0.57	0.61
C(GBR,GW)	0.00	0.10	0.27	0.43	0.55	0.59
C(GBR,GPR)	0.00	0.00	0.04	0.14	0.32	0.43
C(GBR,GCR)	0.00	–0.28	–0.25	–0.13	0.17	0.40
Mean rate of growth of nominal total return on index-linked bonds, GRR						
M(GRR)	7.11	7.05	7.10	7.16	7.05	7.08
SD(GRR)	6.62	5.35	4.49	3.99	2.99	2.14
C(GRR,GQ)	0.60	0.77	0.94	0.98	0.98	0.99
C(GRR,GW)	0.36	0.55	0.81	0.90	0.93	0.94
C(GRR,GPR)	0.03	0.04	0.08	0.22	0.47	0.65
C(GRR,GCR)	0.48	0.17	–0.36	–0.39	0.03	0.38
C(GRR,GBR)	0.00	–0.10	0.22	0.44	0.58	0.64

Table 3.5b TY model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of real growth of wages, JW						
M(JW)	1.99	1.92	1.92	1.94	1.94	1.93
SD(JW)	2.78	2.35	1.78	1.36	0.99	0.70
C(JW,GQ)	–0.53	–0.56	–0.57	–0.60	–0.60	–0.61
Mean rate of growth of real total return on shares, JPR						
M(JPR)	4.66	4.94	4.76	4.80	5.00	5.20
SD(JPR)	16.13	10.12	6.33	4.50	2.84	1.73
C(JPR,GQ)	–0.21	–0.34	–0.51	–0.60	–0.57	–0.49
C(JPR,JW)	0.10	0.20	0.35	0.45	0.51	0.50
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	2.81	2.79	2.88	2.92	2.99	2.93
SD(JCR)	17.76	11.97	8.60	6.18	3.47	2.07
C(JCR,GQ)	–0.47	–0.64	–0.81	–0.84	–0.84	–0.86
C(JCR,JW)	0.26	0.37	0.50	0.55	0.56	0.55
C(JCR,JPR)	0.09	0.24	0.43	0.55	0.56	0.51
Mean rate of growth of real total return on cash, JBR						
M(JBR)	1.01	0.98	0.95	0.96	0.96	0.96
SD(JBR)	3.12	3.22	3.32	3.04	2.34	1.69
C(JBR,GQ)	–1.00	–0.96	–0.91	–0.88	–0.87	–0.86
C(JBR,JW)	0.53	0.54	0.53	0.54	0.54	0.53
C(JBR,JPR)	0.21	0.32	0.47	0.55	0.53	0.43
C(JBR,JCR)	0.47	0.57	0.73	0.78	0.80	0.81
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	2.92	2.95	2.95	2.97	2.94	2.93
SD(JRR)	5.14	3.31	1.52	0.84	0.52	0.36
C(JRR,GQ)	0.12	0.21	0.35	0.27	–0.09	–0.37
C(JRR,JW)	–0.05	–0.11	–0.21	–0.19	0.00	0.20
C(JRR,JPR)	0.02	0.00	–0.15	–0.16	–0.05	0.08
C(JRR,JCR)	0.71	0.53	0.08	–0.04	0.17	0.42
C(JRR,JBR)	–0.12	–0.29	–0.40	–0.25	0.13	0.44

Table 3.6a. Cairns model: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	1.02	1.09	1.33	1.55	1.78	1.95
SD(GQ)	0.50	1.23	2.15	2.53	2.60	2.46
Mean rate of growth of nominal wages, GW						
M(GW)	2.76	3.01	3.22	3.40	3.57	3.73
SD(GW)	2.62	2.25	2.49	2.70	2.71	2.54
C(GW,GQ)	0.22	0.57	0.88	0.95	0.98	0.99
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	5.92	6.99	7.97	8.47	8.77	9.05
SD(GPR)	22.15	14.20	7.49	4.35	2.66	1.90
C(GPR,GQ)	0.00	-0.19	-0.11	0.12	0.46	0.75
C(GPR,GW)	0.01	-0.10	-0.09	0.11	0.45	0.74
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	5.70	5.88	6.27	6.43	6.58	6.57
SD(GCR)	13.18	8.57	4.38	2.30	1.39	1.42
C(GCR,GQ)	-0.01	-0.44	-0.44	-0.21	0.34	0.77
C(GCR,GW)	0.00	-0.25	-0.37	-0.19	0.33	0.76
C(GCR,GPR)	0.74	0.72	0.66	0.60	0.65	0.81
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	3.33	3.34	3.59	3.83	4.07	4.35
SD(GBR)	0.00	1.18	2.16	2.56	2.61	2.45
C(GBR,GQ)	0.00	0.95	0.99	0.99	0.98	0.97
C(GBR,GW)	0.00	0.54	0.87	0.94	0.96	0.96
C(GBR,GPR)	0.00	-0.20	-0.12	0.08	0.41	0.73
C(GBR,GCR)	0.00	-0.46	-0.44	-0.21	0.35	0.79
Mean rate of growth of nominal total return on index-linked bonds, GRR						
M(GRR)	3.59	3.84	4.03	4.21	4.43	4.65
SD(GRR)	2.52	2.15	2.45	2.69	2.68	2.49
C(GRR,GQ)	0.22	0.61	0.92	0.98	0.99	0.99
C(GRR,GW)	0.05	0.35	0.81	0.93	0.97	0.98
C(GRR,GPR)	0.00	-0.11	-0.07	0.14	0.46	0.75
C(GRR,GCR)	0.00	-0.26	-0.39	-0.20	0.34	0.78
C(GRR,GBR)	0.00	0.55	0.88	0.95	0.98	0.99

Table 3.6b Cairns model: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of real growth of wages, JW						
M(JW)	1.74	1.74	1.73	1.74	1.73	1.73
SD(JW)	2.53	1.83	1.15	0.81	0.57	0.40
C(JW,GQ)	0.02	0.02	0.01	-0.01	0.00	0.01
Mean rate of growth of real total return on shares, JPR						
M(JPR)	4.83	5.53	6.17	6.44	6.56	6.84
SD(JPR)	21.94	14.31	7.89	4.71	2.72	1.67
C(JPR,GQ)	-0.03	-0.27	-0.37	-0.44	-0.54	-0.69
C(JPR,JW)	0.01	0.00	0.01	0.00	0.01	-0.01
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	4.68	4.60	4.63	4.56	4.52	4.52
SD(JCR)	13.06	9.06	5.55	3.69	2.48	1.66
C(JCR,GQ)	-0.05	-0.55	-0.71	-0.80	-0.86	-0.85
C(JCR,JW)	0.00	-0.01	0.02	0.01	0.00	-0.01
C(JCR,JPR)	0.74	0.73	0.71	0.70	0.71	0.79
Mean rate of growth of real total return on cash, JBR						
M(JBR)	2.29	2.27	2.25	2.27	2.31	2.37
SD(JBR)	0.51	0.39	0.34	0.41	0.52	0.58
C(JBR,GQ)	-1.00	-0.33	-0.17	-0.16	-0.20	-0.23
C(JBR,JW)	-0.02	0.00	-0.01	-0.02	-0.01	-0.01
C(JBR,JPR)	0.03	0.01	-0.03	-0.10	-0.09	0.14
C(JBR,JCR)	0.05	0.06	0.07	0.11	0.22	0.34
Mean rate of growth of real total return on index-linked bonds, JRR						
M(JRR)	2.57	2.55	2.50	2.53	2.59	2.65
SD(JRR)	2.43	1.68	0.94	0.51	0.28	0.27
C(JRR,GQ)	0.02	0.03	0.06	0.07	-0.02	-0.18
C(JRR,JW)	0.00	0.01	0.01	0.01	0.00	-0.01
C(JRR,JPR)	0.00	0.00	0.04	0.07	0.05	0.19
C(JRR,JCR)	0.00	-0.01	-0.02	-0.05	0.03	0.27
C(JRR,JBR)	-0.02	-0.12	-0.40	-0.47	0.09	0.78

Table 3.7a. Whitten & Thomas: results for nominal returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of inflation, GQ						
M(GQ)	4.03	4.10	4.10	4.26	4.44	4.63
SD(GQ)	3.42	3.27	3.10	2.74	2.17	1.59
Mean rate of growth of nominal wages, GW						
M(GW)	5.75	5.82	5.85	5.95	6.03	6.14
SD(GW)	3.30	3.00	2.71	2.33	1.84	1.36
C(GW,GQ)	0.66	0.79	0.89	0.93	0.94	0.94
Mean rate of growth of nominal total return on shares, GPR						
M(GPR)	13.63	13.34	13.65	14.15	14.52	14.66
SD(GPR)	18.30	12.75	6.85	4.46	3.11	2.22
C(GPR,GQ)	–0.08	–0.11	0.02	0.22	0.42	0.52
C(GPR,GW)	–0.02	–0.04	0.05	0.24	0.42	0.52
Mean rate of growth of nominal total return on long bonds, GCR						
M(GCR)	6.66	6.48	6.39	6.49	6.80	7.10
SD(GCR)	8.71	5.97	3.27	1.82	1.11	1.10
C(GCR,GQ)	–0.29	–0.40	–0.57	–0.57	–0.10	0.47
C(GCR,GW)	–0.19	–0.31	–0.50	–0.52	–0.09	0.43
C(GCR,GPR)	0.37	0.31	0.16	0.05	0.14	0.35
Mean rate of growth of nominal total return on cash, GBR						
M(GBR)	5.49	5.50	5.56	5.70	5.94	6.22
SD(GBR)	0.00	0.57	1.05	1.31	1.44	1.36
C(GBR,GQ)	0.00	0.16	0.29	0.42	0.56	0.67
C(GBR,GW)	0.00	0.13	0.26	0.39	0.52	0.62
C(GBR,GPR)	0.00	–0.07	0.02	0.14	0.28	0.39
C(GBR,GCR)	0.00	–0.29	–0.37	–0.29	0.25	0.71

Table 3.7b Whitten & Thomas: results for real returns from 10,000 simulations.

Term	1	2	5	10	20	40
Mean rate of real growth of wages, JW						
M(JW)	1.71	1.63	1.55	1.49	1.44	1.39
SD(JW)	2.65	1.95	1.34	1.01	0.74	0.53
C(JW,GQ)	–0.46	–0.45	–0.50	–0.56	–0.59	–0.59
Mean rate of growth of real total return on shares, JPR						
M(JPR)	9.23	8.98	9.10	9.19	9.44	9.48
SD(JPR)	18.16	12.99	7.22	4.56	2.87	1.88
C(JPR,GQ)	–0.27	–0.35	–0.42	–0.41	–0.35	–0.29
C(JPR,JW)	0.16	0.22	0.27	0.30	0.27	0.23
Mean rate of growth of real total return on long bonds, JCR						
M(JCR)	2.38	2.19	2.12	2.08	2.17	2.38
SD(JCR)	9.87	7.54	5.37	3.86	2.43	1.41
C(JCR,GQ)	–0.58	–0.72	–0.88	–0.93	–0.90	–0.75
C(JCR,JW)	0.27	0.32	0.44	0.53	0.53	0.43
C(JCR,JPR)	0.44	0.44	0.46	0.46	0.39	0.31
Mean rate of growth of real total return on cash, JBR						
M(JBR)	1.40	1.37	1.48	1.44	1.47	1.55
SD(JBR)	3.32	3.10	2.83	2.36	1.72	1.16
C(JBR,GQ)	–1.00	–0.98	–0.94	–0.88	–0.76	–0.57
C(JBR,JW)	0.46	0.43	0.46	0.48	0.42	0.31
C(JBR,JPR)	0.26	0.33	0.40	0.39	0.30	0.21
C(JBR,JCR)	0.58	0.68	0.78	0.80	0.79	0.76

4 MODIFICATIONS AND EXTENSIONS

4.0 In this section we discuss certain features of stochastic asset models that we have not considered in our investigations above, and have not necessarily implemented or seen implemented; at least, we have not seen public discussion of some of the points. The points we consider are: the use of initial conditions; using what we call "neutralising parameters"; using a "select period" in the early years of a simulation; using alternative formulations for innovations; simulating over shorter intervals; and "hyperising" the models.

4.1 *Initial Conditions*

4.1.1 Each simulation has to start with some set of initial conditions, as discussed in Section 2.8. It is only in a "pure" random walk model that initial conditions have no effect on the results of the simulations, apart from a multiplicative factor for all the indices. In all other cases the choice of initial conditions is important.

4.1.2 The choice is between using neutral initial conditions, defined in one of the ways discussed in Section 2.8, i.e. as the medians or means of the long-run distribution, or as a random sample from that distribution, or alternatively, using what we describe as "market conditions". These might be the actual market conditions on some specific date, or they might be hypothetical non-neutral initial conditions, which are to be investigated to see the effect of changes in the staring point.

4.1.3 Whether one should use neutral or market conditions will depend on the objectives of any simulation exercise. It is of course of theoretical interest to investigate both types of starting point. The only restraint on this is that while one should be able to define one set of neutral initial conditions, or at least limited number of alternatives, an investigation into the effects of all possible starting points becomes limited by the multi-dimensional nature of the problem. There is a very large number of actual past dates that one could use, and if one wished to explore the effect of possible combinations of initial conditions, and the number of initial conditions is at all large, then the multi-dimensional nature of the problem is obvious.

4.1.4 For practical applications, there are advantages on both sides. If one is investigating the appropriate long-run benchmark portfolio for some institution, then neutral initial conditions are appropriate. If the data on which one is working is out of date by the time an investigation is carried out, again neutral initial conditions may be better. Neutral initial conditions can be thought of as approximating to the position if one does not know or is indifferent to, the date on which the investigation is being carried out.

4.1.5 On the other hand, if a stochastic asset model is to be used for tactical asset allocation, then market conditions, as up-to-date as possible, will be appropriate.

Similarly if the model is to be used for pricing of products, and these prices can vary by market conditions, then it will be desirable to use market conditions. Derivative securities are often priced using a different type of stochastic asset model from the actuarial ones we are considering here (with the exception of the Smith and Cairns models, which can also produce risk neutral values which could be used for derivatives pricing). They are often of a much shorter term, with much more frequent time steps, even approximating continuous models. For them, current market conditions, like the current yield curve, may be vital. But just as rates offered by life offices for immediate annuities may well vary with the level of interest rates in the market, so the level of charge for a policy with a fixed guaranteed annuity option should also vary with the level of interest rates in the market.

4.2 *Neutralising the parameters*

4.2.1 An obvious problem about using market initial conditions is that for many models where it makes a difference to the results one can see that the expected returns on assets of different classes may vary enormously with the level of the initial conditions. Thus in a model that includes an autoregressive model for dividend yield, with a fixed mean level for that yield, it is clear that expected returns on shares will be much higher when dividend yields are high, and share prices low, than when the opposite occurs. This runs counter to the theoretical economic idea that, if it were agreed by all that share prices were "too high" or "too low", then the purchases or sales by market participants would soon put them back into balance. It is of interest to discuss whether markets are in fact as rational as they might be expected to be, or whether what may seem to be irrational "bubbles" do or can exist. But we do not follow that line here. Instead we suggest that one way to balance the ideas of both sides is to choose parameter values that correspond to the given market conditions, in such a way that the neutral initial conditions for the chosen set of parameter values is the same as the market conditions at the time.

4.2.2 As a simple example, in the autoregressive model:

$$X(t) = XMU + XA.(X(t-1) - XMU) + XE(t),$$

if one chooses an initial $X(0)$ that is well below XMU, then it is likely that the value of $X(t)$ will rise in the first few years. But if one changes the value of XMU to equal the given $X(0)$, then $X(0)$ becomes the neutral initial condition for the new value of XMU, and there will be no particular bias in the early years of the simulations.

4.2.3 It may not be possible to derive a suitable set of parameters that will match the market conditions in all respects. For example, in the Wilkie model, the neutralising parameter value for QMU, the mean rate of inflation, would be $I(0)$, the rate of inflation over the year before the simulations are to start. But the corresponding neutral values of $DM(0)$ and $CM(0)$, the average rates of inflation over recent past years (exponentially weighted with different factors), are also equal to QMU, and hence to $I(0)$. But the actual values of $DM(0)$ and $CM(0)$ may well not be the same as the actual value of $I(0)$, especially when (as at present in the United Kingdom), inflation is lower than it has been for many years, and has fallen from much higher values in the not too distant past.

4.2.4 A further complication may arise if the neutralising parameters still produce expected returns that seem out of line with what one might imagine a balanced market to be. Thus at present, in many countries, dividend yields are exceptionally low, the rate of inflation is low and the recent rate of growth of dividends in real terms may also have been low. The neutralising parameters of the Wilkie model consistent with this historical data might still produce expected returns for ordinary shares that were lower than the expected returns on bonds, even after neutralising the parameters of the bond model. One may then have to build in the desired risk premium on ordinary shares by choosing a rate of growth of dividends that is large enough. This is consistent with economic theory: if shares at low dividend yields are to be seen as sufficiently attractive, and the market is in equilibrium in some sense (and not in an irrational bubble) then dividends must be expected to rise. However, choosing an appropriate risk premium for ordinary shares would then be a matter of judgement, rather than a consequence of the model.

4.2.5 In spite of these various difficulties we believe that the use of neutralising parameters is worth investigating further. Each model would have to be considered carefully. We have not done this for all the models considered, and we do not quote results in this paper.

4.3 *A select period*

4.3.1 Another possible criticism of a model like the Wilkie one is that the standard deviation of the distribution of certain items over the short run may seem very high. Thus the standard deviation of the forecast for the rate of inflation in the Wilkie model over one year ahead is the same as over any future year. The standard deviation for the "force of inflation" (the continuously compounded rate) is denoted *QSD* and a typical value is 0.04 or so, corresponding to a standard deviation of about 4%, and a two standard deviation range of ±8%. Yet most economic commentators would expect to be able to forecast inflation in the UK over one year ahead to far greater accuracy, probably within ±1.5%.

4.3.2 There is no inconsistency in this. Economic forecasters have available to them a large range of exogenous economic variables, which are not known to the actuarial models. Thus the Wilkie model forecast of inflation is based only on the history of inflation itself, and on no other data. This may be seen as very restrictive feature of that model, but other actuarial models are also restricted to whatever data is simulated in the model. The economists know far more about current economic conditions, and about government or central bank policy than the models do, and so can legitimately make forecasts with much smaller standard deviations, in the short term. But in the further distant future the values of these exogenous variables are unknown. They could be forecast, but only by including them as stochastic features of the model. We understand that many short-term econometric models are very large, involving perhaps hundreds of variables; we have never seen reference to their being used for longer-term simulations.

4.3.3 A way round this is by introducing what actuaries will recognise as a "select period", similar to what is used in mortality investigations. Consider the Wilkie inflation model:

$$I(t) = QMU + QA.(I(t-1) - QMU) + QE(t)$$

where $I(t)$ is the force of inflation for year t, $QE(t) = QSD.QZ(t)$, and QMU, QA and QSD are fixed parameters. At time t=0 we have:

$$I(1) = QMU + QA.(I(0) - QMU) + QE(1),$$

so that $E[I(1)] = QMU + QA.(I(0) - QMU)$ and $\text{SDev}[I(1)] = QSD$. We may use exogenous information to provide "better" estimates of the mean and standard deviation of the forecast of $I(1)$, say $EI1$ and $SDI1$. We can then replace the random variable $QE(1)$, which has mean zero and standard deviation QSD, by a new $QE(1)$, which has a non-zero mean (or bias) equal to $EI(1) - E[I(1)]$, and standard deviation $SDI1$. We can follow through this process in a similar way for other elements of the model and for further years, noting that the adjustments, for example to $QE(2)$, need to take account of the adjustment we have already made to $QE(1)$.

4.3.4 It is probably only worth considering such a select period for very few years (as with mortality investigations); five years at the very most, and probably no more than two; and only for those variables for which serious alternative forecasts are available. Thus, while inflation over one year can perhaps be forecast with greater accuracy than any actuarial model provides, share price changes in the short term seem just as unpredictable as the models represent them to be. Further consideration of the interplay of short term econometric forecasts and actuarial models is worthy of more investigation.

4.4 *Alternative models for innovations*

4.4.1 Each of the models discussed has its own structure of innovations; they are typically normally distributed, though as noted in Section 2.7 the RWV α-stable and Smith models use different distributions, and the Wilkie ARCH model and the Whitten & Thomas model use normal distributions with varying standard deviations.

4.4.2 In principle almost any distribution for the innovations could be combined with almost any model skeleton. Since the distributions of the residuals of many variables in many investigations are leptokurtic (or show fat-tailedness), it may be thought more realistic to model these fat tails. Against this is the argument that, over many periods, unless the innovations are simulated with an α-stable distribution, which has infinite variance, the central limit theorem means that the distribution of the required results approaches normality. However, we do not know how many periods we need before we can ignore the fat-tailedness, and we may well be interested in the results over just a few years, so leptokurtic innovations may well be desirable.

4.4.3 There are at least three ways in which leptokurtic distributions can be modelled. These are:

(a) use a leptokurtic distribution for each innovation (as in the RWV α-stable and Smith);

(b) use an ARCH (autoregressive conditional heteroscedastic) model, in which the standard deviation (or variance) applied to a unit normal innovation varies from year to year within each simulation (as in Wilkie's ARCH model);

(c) use a model with more than one state, so that different values for the standard deviation may apply to the innovations for different years (as Whitten & Thomas).

We discuss these further in turn.

4.4.4 The α-stable distribution, as advocated for example by Finkelstein (1997), Walter (1990), and others, is in some ways the most intellectually satisfying possibility. It can be treated as the discrete sampling of a continuous Lévy process with α-stable increments, of which Brownian motion with normally distributed increments is a special case (with $\alpha = 2$). If the increments over any infinitesimally short period of time, whatever their distribution, have finite variance, then the resulting distribution over longer periods is necessarily normal. Only α-stable processes retain the α-stable property as the sampling interval is increased. Against this, it has been observed that the fat-tailedness of say share price movements is more conspicuous over short intervals, such as days, and much less pronounced as the sampling interval is increased (see, e.g., Campbell, Lo & MacKinlay, 1997, pp 19-21). But to counter this, one could argue that extreme jumps remain possible even over annual intervals, such as 1974 and 1975 in the United Kingdom.

4.4.5 It is relatively easy to simulate α-stable innovations (as in Smith, 1996, who follows Chambers, Mallow & Stuck, 1976; Finkelstein, 1997, also describes their method). However, α-stable distributions must be used with care. To quote standard deviations may be misleading (although we have done it above) and it is really necessary to work with quantiles. They are also not satisfactory for the pricing of derivatives because the prices of simple options are theoretically infinite, and since the prices of variables can apparently "jump" with these innovations such options are not hedgeable. This may, however, be a realistic assessment of the options market; we shall have to wait and see.

4.4.6 Other leptokurtic finite variance distributions include:

Smith's difference of two gamma-distributed variables, with appropriate parameter values; in fact Smith uses four gamma variates, with different parameters, and constructs linear combinations of these four, with different weights, before taking differences, but there are clearly many variations of this concept;

the differences between two lognormal variables, either with the same or with different parameter values; or differences between any two variables with a similarly skew distribution, such as the many used in general insurance work, Weibull, Pareto, Burr, etc; again, we have not seen these used for simulation work, nor described as applying to real data;

t-distributed variables; these were suggested by Praetz (1972), but we have not seen them used in any simulation work;

other fat-tailed distributions, such as described by Johnson & Kotz (1970); we have not investigated these ourselves; a problem about using a less familiar distribution is that a satisfactory method for simulating random variates from such a distribution may not be available; for this reason the differencing approach described above may be the most convenient.

4.4.7 ARCH models have come into prominence in recent years, particularly for modelling changing volatility in option pricing models. Wilkie uses a simple model for the standard deviation of his inflation model. We simplify his notation in what follows. Each innovation $e(t)$ is derived as the product of a unit normal random variate, $z(t)$, and a standard deviation that varies with time, $\sigma(t)$, which is the square root of the variance $V(t)$. $V(t)$ depends in Wilkie's model only on the previous deviation, at time $t-1$, of the variable being simulated, $X(t-1)$, from its mean:

$$V(t) = a^2 + b.(X(t-1)-\mu)^2$$

Thus the variance is large when $X(t-1)$ deviates from its mean position in either direction by a large amount, and *vice versa* when the deviation is small. This formulation has the advantage that the dependency is explicit; the value of $X(t-1)$ is known.

4.4.8 Alternative formulations could bring in other items, for example:

$$V(t) = a^2 + b.(X(t-1)-\mu)^2 + c.e(t-1)^2 + d.V(t-1) + \eta(t),$$

where $\eta(t)$ is another innovation with some standard deviation, σ_η. Thus the variance could be made to depend on the previous value of the simulated variable, the value of the previous innovation, the previous value of the variance itself, and another innovation that applies to the variance. Each of the dependencies could be pushed back further to include relationships with values at time $t-2$, and with $\eta(t-1)$, but the more complex the model the more difficult it is to parameterise satisfactorily from past data, and while much attention is paid to short-term measures of changing volatility, we have seen no work done (other than Wilkie, 1995, which only uses one model for one variable) on longer-term models.

4.4.9 Models which assume more than one "state" can be formulated in a variety of ways. One is the threshold model of Tong (1990) which has been used by Whitten & Thomas. Their model is a development of the Wilkie model, in which there are two states, which apply to all variables together. Different sets of parameters apply in each state, so in principle there are twice as many parameters as in the Wilkie model (though a few have the same values in both states). Which state one is in for each year of each simulation in respect of inflation is determined by whether the value of the force of inflation in the previous year, $I(t-1)$, is less than or greater than 0.1, and in respect of the other variables on whether the value of the force of inflation in the current year, $I(t)$ is less than or greater then 0.1, i.e. the investment world is assumed to behave differently when inflation is greater than about 10% per year than when it is less.

4.4.10 It would be possible with such a model to have more than two states, with more than one threshold; it would also be possible to apply different states to different variables, rather than one threshold determining all variables, but Whitten & Thomas are satisfied with what seems to be one of the simplest threshold models. However, the effect of using different values for the standard deviations in different years, even though the innovations in each state are normally distributed, is to make the innovations overall leptokurtic.

4.4.11 An alternative method of introducing two (or more) states has been described by Harris (1999). In his method there also two states, each with its own set of parameters. Which state applies in each year of each simulation is controlled by a Markov chain. For each state there is a prescribed probability of changing to the other state each year. Thus if the simulation is in state 1 in year $t-1$ there is a probability denoted p_1 of moving to state 2 in year t; and a probability of $1-p_1$ of staying in state 1; likewise if the simulation is in state 2 in year $t-1$ there is a probability denoted p_2 of moving to state 1 in year t; and a probability of $1-p_2$ of staying in state 2. This has similar consequences to the threshold model in relation to the fat-tailedness of the innovations overall but implements it in a different way. Harris does not provide a model that is of direct investment use; rather his published paper provides a methodology for deriving such a model.

4.4.12 A further way of introducing picking from two normal distributions is to use, for each innovation, a mixture distribution. One could pick first from a normal distribution with a "normal" size of standard deviation. One could then pick either from a Bernoulli distribution ($k = 0$ or 1 with probability p), or from a Poisson distribution (an integer $k \geq 0$, with parameter λ); one could then make k pickings from a normal distribution with a larger standard deviation and add these to the first normal; or one could replace the first normal with the sum of the pickings if $k > 0$. We have seen no descriptions of investment models that implement this method, though we have used it ourselves in a different type of application. The method is capable of as much elaboration as one wishes, and it resembles Harris' method in some respects. But note that only the innovation is affected; the skeleton of the model remains unchanged.

4.5 *Simulating over shorter intervals*

4.5.1 All the models, except that of Cairns and Smith, are designed to be simulated with annual steps. At times one might like to simulate over shorter intervals. Monthly intervals might be appropriate for considering unit-linked life assurance policies with monthly premiums. Much shorter intervals would be needed if one were to use the models for derivative pricing. In some cases it is possible to find an equivalent model for shorter periods, or even a continuous one. For example, the RWV lognormal model easily converts to a model with continuous Brownian motion, and the RWV α-stable model to one driven by a Lévy-stable process. A first order autoregressive model is equivalent to a continuous Ornstein-Uhlenbeck process. Each of these parts therefore can easily be made to fit any frequency.

4.5.2 Other parts of models do not convert so easily. The Wilkie inflation model deals with inflation explicitly over the previous year, and this does not easily convert to a model with shorter intervals.

4.5.3 A possible approach, however, which could be applied to any model, is to simulate first at annual intervals, and then to fill in the intervening (say) months by means of "Brownian bridges", that is by simulating a random walk over each year conditional on finishing at the already simulated next annual value. One needs only to choose a monthly standard deviation for each variable to be simulated. This could be chosen as the monthly equivalent of the annual standard deviation (by multiplying by $1/\sqrt{12}$), or in some other way. We have not seen reference to this approach having been implemented.

4.5.4 The Smith and Cairns models, however, are defined in continuous terms throughout, and can be implemented with any desired frequency, though with more frequent intervals they may take much longer to run.

4.6 *Hyperising the models*

4.6.1 A valid criticism of almost all of the models under consideration (and including a pure random walk model) is that, over a sufficiently long period, the average rate of growth of any index tends towards the mean rate of growth implied by the fixed parameters of the model; this is a consequence of the law of large numbers. Only the α-stable models escape this criticism, but then the concept of an "average" with such a model is doubtful; only medians have much meaning; but the median rate of growth then seems to behave in the same way as the average does for other models.

4.6.2 This feature is a consequence of assuming fixed parameter values in the models, which are assumed to be known. A way round this is to use what Bayesian statisticians are familiar with and call a "hypermodel". In such a hypermodel it is assumed that the parameters of the basic model themselves are random variables with some joint distribution function. Thus, instead of using a fixed value of *QMU* in the Wilkie inflation model, one could assume that the value of *QMU* for simulation *s*, denoted *QMU*(*s*), is picked from a normal distribution with a mean *QMUMU* and a standard deviation *QMUSD*. A different value of *QMU*(*s*) would be picked for each simulation.

4.6.3 Other parameters could be picked likewise. In the first place one could consider picking from independent normal distributions for each parameter, but one would quickly think of using a multivariate normal distribution with correlations between the parameters. These might be based on the information matrix if the parameters have been estimated by maximum likelihood, or on the corresponding Bayesian statistics. Then some of the parameters should not be thought of as normally distributed, except approximately. Variances are better described as being gamma (χ^2) distributed, and autoregressive parameters such *QA*, which are limited to the open set (0,1), i.e. excluding the ends, as being beta distributed. A complication is that methods of introducing correlation between variables distributed differently may not be easy.

4.6.4 If normal distributions are used as an approximation, one would have to check that variances (or standard deviations) were restricted to being positive, and that autoregressive parameters were restricted to the range [0,1], i.e. possibly allowing 0 and 1.

4.6.5 This general approach was described and implemented by Wilkie (1986), though the value of each parameter was chosen independently. The concept does not seem to have been developed since then.

4.6.6 The Monte Carlo Markov Chain methods described by Harris (1999) are Bayesian in approach, and provide an empirical multivariate distribution for the set of parameters being estimated, with say 10,000 sets of possible parameter values. However, if the number of parameters is at all large, this becomes an extremely large data set to handle. Further, it seems to us much harder to adjust the hyperparameters of such an empirical distribution, in the same way that one can easily adjust the values of say *QMUMU* and *QMUSD* noted above.

4.6.7 A further complexity could be to combine this approach with the multi-state method described by Harris. Within each simulation one could change parameter sets each year with a quite small probability. This would be intended to represent changes in the overall economic environment. Whether the resulting complexity would produce a model that was capable of interpretation remains to be seen.

4.7 *Other developments*

4.7.1 In this section we have restricted ourselves to commenting on features that could be introduced into any model similar to those we have considered in this paper. We have not discussed alternative approaches to the skeletons of models, though these are possible and desirable in the pursuit of knowledge. Including further simulated variables, such as Gross National Product, are possibilities that have been discussed elsewhere. We await the results of investigations and new models on these lines.

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